Continuum Limit of the Ising model

It will be extremely helpful to consider a continuum limit of the Ising model. It will resemble Landau theory of a fluctuating variable.

The simplest way to proceed is to just argue its form based on symmetries, similar to the Landau theory. We replace a configuration $\{s\}$ of Ising spins by a <u>function</u> $\varphi(\vec{r})$ where the Ising sym. is implemented as $\varphi(\vec{r}) \to -\varphi(\vec{r})$. The sum over all configurations is replaced by the functional integral over all functions $\varphi(\vec{r})$.

Therefore, we write:

$$Z = \operatorname*{Tr}_{\{s\}} e^{J\sum_{\langle ij\rangle} s_i s_j + h\sum_i s_i}$$

$$\approx \int D\varphi(\vec{r})\; e^{-\int d^d r\, \mathcal{L}(\varphi(\vec{r}))}$$

functional integral.

where $\mathcal{L}$ is the 'Lagrangian' corresponding to a given f^n $\varphi(\vec{r})$.

To determine $S(\varphi(\vec{r}))$, we only demand that $S(\varphi(\vec{r}))$ is (a) local (b) respects all the symmetries of the Ising model, i.e., the Ising sym. and the lattice symmetries (translation, rotation etc.). First, consider the special case where $\varphi(\vec{r})$ is independent of $\vec{r}$. If so, then $\int d^d r \, \mathcal{L} = \int d^d r \, [t\varphi^2 + u\varphi^4 + h\varphi]$ $= V\,[t\varphi^2 + u\varphi^4 + h\varphi]$ where $V = \int d^d r$ is the total volume of the system. In this special case, one may now evaluate the partition f^n Z, using saddle point approximation and one recovers the Landau theory. However, in reality, there are additional terms allowed by symmetry since φ is a f^n of $\vec{r}$ in general. The first few terms are

$$\mathcal{L} = \Big[\tfrac{1}{2}(\vec{\nabla}\varphi)^2 + t\varphi^2(r) + u\varphi^4(r) + h\varphi(r) + v\varphi^3(r) + \dots\Big]$$

Note that if the external magnetic field were zero, then both $h = v = 0$ due to $\varphi \to -\varphi$ sym.

It turns out that one can set $v = 0$ without any loss of generality. This is because in the absence of Ising sym. (when v and h are allowed), one is allowed to perform a uniform shift:

$\varphi(r) \rightarrow \varphi(\vec{r}) + c$ where c is a constant.

This extra degree of freedom 'c' can now be suitably chosen so that the coefficient of the v^3 term is zero. Therefore, we will drop this term in our discussion from now on.

One way to 'derive' the above form of $\mathcal{L}(\varphi(\vec{r}))$ starting from the Ising model is to relax the constraint $s(r) = \pm 1$ in the Ising model and impose it softly / energetically via a term

$(s^2(r) - 1)^2$.

Further, $-J \sum_{\langle rr' \rangle} s(r)\, s(r') \simeq$

$$-J \quad s(\vec{r}) \left[s(\vec{r}) + (\vec{r} - \vec{r}\,') \cdot \vec{\nabla} s(\vec{r}) + \frac{1}{2} (\vec{r} - \vec{r}\,')^2 \, \nabla^2 \vec{s}(\vec{r}) + \cdots \right]$$

For example, on a square lattice nearest-neighbor model, the first few terms are

$$-J\, s^2(\vec{r}) - J\left[s\, \partial_x s + s\, \partial_y s\right] - \frac{J}{2}\left[s\, \partial_x^2 s + s\, \partial_y^2 s\right]$$

$$\Rightarrow H \simeq \int dx\,dy \Big\{ -J\, S^2(\vec{r}) - J\,[S\,\partial_x S + S\,\partial_y S] - \frac{J}{2}\,(S\,\partial_x^2 S + S\,\partial_y^2 S) + u\,(S^2(\vec{r}) - 1)^2 \Big\}$$

Integrating by parts, $\int dx\,dy\,[S\,\partial_x S + S\,\partial_y S]$ $= 0$ while $-\int dx\,dy\; S\,(\partial_x^2 S + \partial_y^2 S)$ $= \int dx\,dy\,(\vec{\nabla} S)^2$. Thus, putting everything together, one obtains the same functional form as the one argued based on symmetry

$$\mathcal{L} = \int d^d r \left[\frac{(\nabla \varphi)^2}{2} + t\,\varphi^2 + u\,\varphi^4 + h\,\varphi \right]$$

where we have rescaled $S \propto \varphi$ so that the coefficient of $\frac{(\nabla\varphi)^2}{2}$ term is unity. Further, there is no need to relate the parameters t, u etc. to the original parameters J, h since all these will anyway flow under RG. The only constraint is symmetry: t and u are coeffs. of Ising symmetric terms and therefore can only be an even f^n of the original magnetic field h.

This also holds true for the RG flow: the flow of t, e.g. can't contain a term odd under Ising sym. e.g. $\frac{dt}{dl} \sim h$ cannot occur while $dt/dl \sim h^2$ can. Similarly $\frac{dh}{dl} \sim ht$ can occur in principle but $\frac{dh}{dl} \sim t$ can't.

Thus, RG equations will break into even and odd subspaces.
